

Resampling Based Empirical Prediction: An Application to Small Area Estimation

BY SOUMENDRA N. LAHIRI¹, TAPABRATA MAITI¹, MYRON KATZOFF², AND VAN PARSONS²

¹*Department of Statistics, Iowa State University, Ames, IA 50011; snlahiri,taps{@iastate.edu}*

²*NCHS/CDC, 3311 Toledo Road, Hyattsville, Maryland 20782; mjk5, vlp1{@cdc.gov}*

SUMMARY

Best linear unbiased prediction is well known for its wide range of applications including small area estimation. While the theory is well established for mixed linear models and under normality of the error and mixing distributions, the literature is sparse for nonlinear mixed models under nonnormality of the error or of the mixing distributions. This article develops a resampling based unified approach for predicting mixed effects under a generalized mixed model set up. Second order accurate nonnegative estimators of mean squared prediction errors are also developed. Given the parametric model, the proposed methodology automatically produces estimates of the small area parameters and their MSPEs, without requiring explicit analytical expressions for the MSPE.

Some key words: Best predictor; Bootstrap; Kernel; Mean squared prediction error.

1 INTRODUCTION

Small area estimation (SAE) is an important statistical research area due to its growing demand from public and private agencies. The variance of a small area estimator based on the direct small area sample is unduly large and hence, there is a need for constructing model based estimators with low mean squared prediction error (MSPE). A good account of small area estimation research is available in a recent book by J.N.K. Rao (Rao, 2003). Although, in theory, it is possible to use such a model based approach, in practice a statistician often faces some challenging problems in implementing it due to the fact that for each model, estimators must be derived and their properties studied. Indeed, a small deviation from the

standard model assumptions may require a considerable amount of analytical work and need special expertise. For example, Prasad and Rao (1990) (hereafter referred to as PR) derived small area estimation formulas assuming normality of both the sampling distribution and the population distribution (for two-level small area models, discussed later) and with the moment based estimators of model parameters. After about a decade, Datta and Lahiri (2000) extended this approach when the model parameters are estimated by the maximum likelihood approach. Recent works of Jiang, Lahiri and Wan (2002) and Lahiri and Maiti (2003) (hereafter referred to as JLW and LM, respectively) allow a more general framework, but both works require the knowledge of the exact functional forms of the MSPE, which are known only in few simple cases. However, a general solution to finding the best estimator of the small area parameters or of its functions, and estimation of the associated MSPE are not available. A second problem with the existing approaches (except for the LM method) to estimating the MSPE is that these methods do not always produce non-negative estimates. Though the linearization technique of PR produces non-negative estimates under normality, the jackknife method may produce negative MSPE estimates (Bell, 2002). Consequently, there is a great demand for a general estimation system where the user can only specify the distributions and then valid estimates of the small area parameters and their MSPEs can be obtained without much of analytical efforts.

In this paper, we consider a general two level aggregate data model and develop a unified system for prediction of small area parameters and estimation of the associated MSPE. Here we extend the “perturbation” or “tilting” method of LM and construct a nonnegative estimator of the MSPE that achieves second order accuracy for bias correction without requiring explicit analytical derivation of the MSPE function. The key idea is to combine the LM approach with the parametric version of the bootstrap method of Efron (1979) so that accurate numerical approximations to various intermediate population quantities can be generated numerically. We show that under some regularity conditions, the proposed MSPE estimator attains second order accuracy for a wide range of parametric distributions and for a general class of model parameter estimates and their nonlinear functions, without requiring the user to derive the formulas on a case by case basis.

The rest of the paper is organized as follows. In Section 2, we consider the general two level aggregate data model that is commonly used in the context of small area estimation. In Section 3, we describe the (estimated) best predictor of functions of the small area parameters. In Section 4, we briefly describe the existing approaches to MSPE estimation and also give a description of the proposed method. Theoretical properties of the proposed method are given in Section 5. Results from a simulation study and some concluding remarks are presented in Sections 6 and 7, respectively. Proofs are given in the Appendix.

2 GENERALIZED MIXED MODELS FOR SMALL AREA ESTIMATION

Consider the general two level aggregate data model

$$y_i|\theta_i \overset{ind}{\sim} F_1(\cdot; \theta_i, R_i), \quad \theta_i \overset{ind}{\sim} F_2(\cdot; x_i, \lambda, G_i), \quad i = 1, \dots, m, \quad (2.1)$$

where, R_i and G_i are known functions of a vector of p -parameters $\psi = (\psi_1, \dots, \psi_q)$, say, $(R_i, G_i) = g_i(\psi)$. Thus, the model is determined by the parameter vector $\delta \equiv (\lambda^T, \psi^T)^T$, a $(p+q) \times 1$ vector of constants. Usually, y_i 's are direct survey estimators with sampling variance R_i , θ_i 's are small area parameters and x_i a set of covariates available at the estimation stage. Aggregate and generalized linear mixed effects models are special cases of (2.1).

Consider the Fay-Herriot (1979) type small area model

$$y_i = \theta_i + e_i, \quad \theta_i = x_i^T \lambda + v_i \quad (2.2)$$

where e_i 's are independent $N(0, s_i)$ with known s_i , v_i 's are iid $N(0, \sigma_v^2)$ and e_i and v_i 's are independent. Furthermore, x_i is a known $p \times 1$ vector of co-variates, λ is the vector of regression coefficients; y_i is the direct survey estimator of θ_i . Note that (2.2) can be written as $y_i = x_i^T \lambda + v_i + e_i$ which is a special case of a linear mixed model where both F_1 and F_2 are normal cdf.

Next consider the mixed logistic model, where conditional on small area parameter p_i , the direct estimator y_i is binomial (n_i, p_i) , $i = 1, \dots, m$; here n_i is the number of sampled units in the i -th small area. Then, consider the model

$$\theta_i \equiv \text{logit}(p_i) = x_i^T \lambda + v_i, \quad (2.3)$$

where the v_i 's are iid $N(0, \sigma_v^2)$. In this case F_1 is binomial and F_2 is normal in the logit scale. This is a special case of generalized linear mixed model.

Our objective is to make inference about a function of the small area parameter θ_i

$$\beta_i = h(\theta_i), i = 1, \dots, m, \quad (2.4)$$

where h is a suitable function chosen by the user. For example, the ‘‘Small Area Income and Poverty Estimation’’ (SAIPE) project of the US Census Bureau uses the log value of the direct estimates for estimating poverty at the county level and thus an inverse transformation needed for the parameter of interest. We would like to emphasize that, at the second level of modeling, the structure always need not be of the form $h(\theta_i) = x_i^T \lambda + v_i$. In fact, we can also use nonlinear modeling, such as $h(\theta_i) = \kappa(x_i; \lambda, v_i)$. where κ is a nonlinear function.

3 DEVELOPMENT OF THE BEST AND EMPIRICAL BEST PREDICTORS

As an estimator of the small area parameter, we will take the best predictor (BP) as defined below. Let $\beta_i = h(\theta_i)$ be the parameter of interest. We define the BP and the *empirical best predictor* (EBP) of β_i , respectively, by

$$\tilde{\beta}_i = E_{\delta}\{h(\theta_i)|y\}, \quad (3.1)$$

$$\hat{\beta}_i = E_{\hat{\delta}}\{h(\theta_i)|y\}, \quad 1, \dots, m, \quad (3.2)$$

where $\hat{\delta}$ is an estimator of δ . For example, in the Fay-Herriot model (2.2), the BP of $h(\theta_i) = \theta_i$ takes the form $\tilde{\beta}_i = x_i^T \lambda + \frac{\sigma_v^2}{\tau_i}(y_i - x_i^T \lambda)$, where $\tau_i = \sigma_v^2 + s_i$. For a general $h(\cdot)$, however, a closed form simple expression for the BP/EBP and their MSPE may not be available. Consequently, the PR-type SAE methodology based on Taylor's expansions may not be readily applicable.

Next, we derive some useful general formulas for the EBP of (3.2). Note that by the independence of y_i 's, the conditional distribution of θ_i given y_1, \dots, y_n depends only on y_i (and δ). Hence, $\tilde{\beta}_i = E_{\delta}\{h(\theta_i)|y\} = \int h(t)F_{\theta_i|y_i}(dt; \delta) \equiv \xi_i(y_i; \delta)$ say, where $F_{\theta_i|y_i}(\cdot; \delta)$ denotes the conditional distribution of θ_i given y_i . The EBP is given by

$$\hat{\beta}_i = \xi_i(y_i; \hat{\delta}). \quad (3.3)$$

First consider the case where the marginal distribution of θ_i has a probability density function (pdf) $f_2(\cdot; x_i, \lambda, G_i)$ (with respect to the Lebesgue measure) and the conditional distribution $F_1(\cdot; \theta_i, R_i)$ of y_i given θ_i has a generalized density $f_1(\cdot; \theta_i, R_i)$ (i.e., the Radon-Nikodym derivative with respect to a σ -finite measure). For example, f_1 can itself be a pdf or a probability mass function (pmf) for a discrete probability distribution. In this case, the EBP is given by

$$\hat{\beta}_i = \xi_i(y_i; \hat{\delta}) = \frac{\int h(t) p_i(y_i, t; \hat{\delta}) dt}{\int p_i(y_i, t; \hat{\delta}) dt}, \quad (3.4)$$

where $p_i(y, t; \delta) = f_1(y; t, R_i) f_2(t; x_i, \lambda, G_i)$. Next consider the case where the marginal distribution of θ_i is discrete and has a pmf $f_2(\cdot; x_i, \lambda, G_i)$ and $F_1(\cdot; \theta_i, R_i)$ has a generalized density $f_1(\cdot; \theta_i, R_i)$ as above. Here the EBP is given by

$$\hat{\beta}_i = \xi_i(y_i; \hat{\delta}) = \frac{\sum_t h(t) p_i(y_i, t; \hat{\delta})}{\sum_t p_i(y_i, t; \hat{\delta})}, \quad (3.5)$$

where $p_i(y, t; \delta)$ is as before and where the sum in (3.5) runs over all t in the support of θ_i . In many applications, formulas (3.4) and (3.5) can be implemented using numerical methods, e.g., numerical integration, MCMC, importance sampling, etc. For example, for the logit-normal model with the canonical link, $\xi_i(y_i; \delta) = [\int \alpha_i(t)(y_i + 1)\{1 + \alpha_i(t)\}^{-n_i-1} \phi(t) dt] / [\int \alpha_i(t) y_i \{1 + \alpha_i(t)\}^{-n_i} \phi(t) dt]$, where $\alpha_i(z) = \exp(x_i^T \lambda + \sigma_v z)$ and ϕ is the $N(0,1)$ pdf (e.g., see, McCulloch and Searle (2001, pp 273) and JLW). In this case, the EBP can be easily evaluated by generating $N(0,1)$ variates and using the Monte-Carlo method.

Remark 1: (*Parameter estimation*). In general, the maximal likelihood estimates (MLE's) do not have any closed form expressions. Except for the conjugate and linear link models, maximization of the marginal likelihood involves integration with respect to the distribution function F_2 . There is no unique way of evaluating this integral. Using advanced techniques such as EM based MLE, Markov Chain Monte Carlo (MCMC) based MLE, etc., the MLE's can be obtained for a large class of distributions. An excellent account of guidelines for the general mixed linear models can be obtained in Chapter 10 of McCulloch and Searle (2001). We mention that the SAE methodology developed here is equally applicable for other type of parameter estimators such as those based on method of moments or estimating equation approaches, provided they are $m^{1/2}$ consistent.

Remark 2: For situations where a direct implementation of (3.4) or (3.5) is difficult, we now describe some approximations to the EBP using the bootstrap method of Efron (1979) and the nonparametric functional estimation methodology. Note that $\hat{\beta}_i$ is the conditional expected value of a function of θ_i for fixed δ evaluated at $\delta = \hat{\delta}$. This suggests that under mild regularity conditions, we may approximate $\hat{\beta}_i$ to any desired level of accuracy by using standard regression function estimation methods, such as Nadaraya-Watson estimators, local polynomial estimators, etc. Let $\{y_i^{*j}, \theta_i^{*j}\}_{j=1}^J$ be generated values using model (2.1), but with $\delta = \hat{\delta}$. When the distributions of θ_i and y_i are continuous, we propose a Nadaraya-Watson approximation to $\hat{\beta}_i$, given by

$$\hat{\beta}_i^* = \frac{\sum_{j=1}^J k_{(\cdot)}(y_i^{*j} - y_i) h(\theta_i^{*j})}{\sum_{j=1}^J k_{(\cdot)}(y_i^{*j} - y_i)}, \quad (3.6)$$

where $k_{(\cdot)}$ is a symmetric kernel function chosen suitably. There are many choices of $k_{(\cdot)}$, such as a Gaussian kernel $k_{(\cdot)}(x) = \frac{1}{b}k(x/b)$ where b is the bandwidth and $k(x) = \phi(x)$, the standard normal density function. On the other hand, when the marginal distribution of y_i is discrete, we propose

$$\hat{\beta}_i^* = \frac{\sum_{j=1}^J h(\theta_i^{*j}) I_{[y_i^{*j}=y_i]}}{\sum_{j=1}^J I_{[y_i^{*j}=y_i]}}, \quad (3.7)$$

where $I_{[\cdot]}$ denotes the indicator function. Results on Nadaraya-Watson estimators of regression functions imply (cf. Härdle (1991)) that

$$|\hat{\beta}_i - \hat{\beta}_i^*|^2 = O\left((Jb)^{-1} + b^{-2}\right) \quad \text{in probability,} \quad (3.8)$$

as $J \rightarrow \infty$ and $b \rightarrow 0$ in such a way that $Jb \rightarrow \infty$. The bound in (3.8) is available uniformly over $i = 1, \dots, m$, provided there exists a constant $C \in (0, \infty)$ such that $E_{\delta}|\xi_i''(Y_i; t)| + E_{\delta}|g_i''(Y_i; t)| < C$ for all $i = 1, \dots, m$ and for all $t \in \mathcal{N}$, a neighborhood of the true value of the unknown parameter δ . Here, $\xi_i''(y; t) = \frac{\partial^2}{\partial y^2} \xi_i(y; t)$, $g_i''(y; t) = \frac{\partial^2}{\partial y^2} g_i(y; t)$, and $g_i(y; t)$ is the marginal density of Y_i . For the discrete case, a direct computation shows that

$$|\hat{\beta}_i - \hat{\beta}_i^*|^2 = O\left(J^{-1}\right) \quad \text{in probability,} \quad (3.9)$$

as $J \rightarrow \infty$. This bound is also available uniformly in i , provided $E_{\delta}|h(\theta_i)|^2 < C$ for all i and for all $t \in \mathcal{N}$, where $C \in (0, \infty)$ is a constant, and \mathcal{N} is as above.

Thus, for both the discrete and the continuous data, the accuracy of the approximation $\hat{\beta}_i^*$ to $\hat{\beta}_i$ increases with larger values of J . For the continuous case, we need to specify a choice of the bandwidth b . For kernels arising from symmetric probability densities, the optimal choice of b is of the order $J^{-1/5}$. We take the bandwidth b of this optimal order, e.g., $b = J^{-1/5}$, and attain a desired level of accuracy by choosing J suitably large. Finite sample accuracy of the approximations (3.6) and (3.7) are typically very good. See Table 1 in Section 6 below which reports the relative biases and MSPE'S of (3.6) and (3.7) for the normal-normal and the logit-normal examples.

4 MEAN SQUARED PREDICTION ERROR AND ITS ESTIMATION

4.1 Background

As a measure of accuracy of the EBP $\hat{\beta}_i$, we shall consider the Mean Squared Prediction Error (MSPE) of $\hat{\beta}_i$, $MSPE(\hat{\beta}_i) = E_\delta(\hat{\beta}_i - \beta_i)^2 \equiv M_i(\delta)$. It is easy to show that

$$M_i(\delta) = E_\delta(\tilde{\beta}_i - \beta_i)^2 + E_\delta(\hat{\beta}_i - \tilde{\beta}_i)^2 \equiv M_{1i}(\delta) + M_{2i}(\delta), \quad \text{say.} \quad (4.1)$$

The first term $M_{1i}(\delta)$ is the mean squared error of the (ideal) best predictor $\tilde{\beta}_i$ while the second term $M_{2i}(\delta)$ accounts for the extra variability due to the estimation of δ . Typically,

$$M_{1i}(\delta) = O(1) \quad \text{and} \quad M_{2i}(\delta) = O(m^{-1}) \quad \text{as} \quad m \rightarrow \infty. \quad (4.2)$$

It is tempting to plug in $\hat{\delta}$ in (4.2) and get a simple MSPE estimate as

$$mspe_{\text{SIM}}(\hat{\beta}_i) = M_{1i}(\hat{\delta}) + M_{2i}(\hat{\delta}). \quad (4.3)$$

However, this approach has two drawbacks. First, explicit expressions for the functions $M_{1i}(\delta)$ and $M_{2i}(\delta)$ are not always available. In the very special case of the normal-normal Fay-Herriot model, an expression for $M_{1i}(\delta)$ and an approximation for $M_{2i}(\delta)$ are available for $h(\theta_i) = \theta_i$, $i = 1, \dots, m$. Even for this model, expressions are not available for a *nonlinear* function of θ_i and one has to derive those. For example, Slud and Maiti (2006) derived the expressions for MSPE estimates under normal set up when h is an exponential function.

The second problem with the above approach is a little more subtle. To describe it, note that typically, the estimator $\hat{\delta}$ has bias and variance of order $O(m^{-1})$, which propagate through the simple MSPE estimator, leading to $E\{M_{1i}(\hat{\delta})\} = M_{1i}(\delta) + O(m^{-1})$ and $E\{M_{2i}(\hat{\delta})\} = M_{2i}(\delta) + o(m^{-1})$ as $m \rightarrow \infty$. (Here and in the following, we often drop the subscript δ to ease notation). Thus, $E\{M_{1i}(\hat{\delta}) - M_{1i}(\delta)\}$, the bias of the simple estimator of $M_{1i}(\delta)$, is of the order $O(m^{-1})$ which masks the contribution of $M_{2i}(\cdot)$ to the MSPE of $\hat{\beta}_i$ (cf. (4.2)).

In view of the second problem, in the SAE literature, it is customary to require that the bias of a “good” estimator of $\text{MSPE}(\hat{\beta}_i)$ be of *smaller order than* $O(m^{-1})$. Traditionally, the bias of the naive estimator $M_{1i}(\hat{\delta})$ is reduced by explicit bias correction, either by using a Taylor’s expansion of the function $M_{1i}(\cdot)$ (cf. PR) or using the Jackknife method (cf. JLW). Other related work include Pfeiffermann and Tiller (2005) and Pfeiffermann and Glickman (2004). The first paper approximated $M_{2i}(\cdot)$ and the bias of $M_{1i}(\hat{\delta})$ under a state space model based on parametric bootstrap, assuming normality of the errors. The second used a bias corrected estimator of $M_{1i}(\cdot)$ and a parametric bootstrap estimator of $M_{2i}(\cdot)$, for the Fay-Herriot model. Pfeiffermann and Glickman also developed a ‘nonparametric’ bootstrap method that did not require generating samples from a distribution. Nonetheless, normality was still assumed implicitly. In a recent work, LM proposed a new approach to bias correction that attains second order accuracy and at the same time, produces a nonnegative estimator of the MSPE. Here, we extend the LM approach to the case of estimating the MSPE of a general function of θ_i with second order accuracy under a general two-level parametric model, even when exact expressions for the functions $M_{1i}(\cdot)$ and $M_{2i}(\cdot)$ are not available.

For completeness, we now briefly describe the LM method. Suppose that for $i = 1, \dots, m$,

$$\sum_{j=1}^k |M_{1i}^{(j)}(\delta)| > \epsilon_0, \quad (4.4)$$

for some $\epsilon_0 > 0$, where for a smooth function $f : \mathbb{R}^k \rightarrow \mathbb{R}$, $f^{(j)}$, $f^{(j,r)}$ and $f^{(j,r,s)}$ denote the first, the second and the third order partial derivatives with respect to the j -th co-ordinate, the (j, r) -th co-ordinates, and the (j, r, s) -th co-ordinates, respectively, $j, r, s = 1, \dots, k$, where k is the number of model parameters. Condition (4.4) says that $M_{1i}^{(j)}(\delta) \neq 0$ for some j . For notational simplicity, we suppose that $M_{1i}^{(1)}(\delta) \neq 0$. Then, the *preliminary*

perturbed estimator of δ for the i -th small area is defined as $\bar{\delta}_i = \hat{\delta} - \hat{B}_i \{M_{1i}^{(1)}(\hat{\delta})\}^{-1} e_1$ where $\hat{B}_i \equiv \sum_{j=1}^k M_{1i}^{(j)}(\hat{\delta}) \hat{b}(j) + \frac{1}{2} \sum_{j=1}^k \sum_{r=1}^k M_{1i}^{(j,r)}(\hat{\delta}) \hat{V}(j, r)$, with $\hat{b} = (\hat{b}(1), \dots, \hat{b}(k))$ and $\hat{V} = ((\hat{V}(j, r)))_{k \times k}$ respectively denoting some suitable estimators (e.g., bootstrap estimators) of the bias and the variance of $\hat{\delta}$, and $e_r \in \mathbb{R}^k$ has 1 in the r -th position and zeros elsewhere, $1 \leq r \leq k$. The LM estimator of the MSPE is now defined as

$$mspe_{\text{LM}}(\hat{\beta}_i) = M_{1i}(\check{\delta}_i) + M_{2i}(\hat{\delta}), i = 1, \dots, m, \quad (4.5)$$

where $\check{\delta}_i$ is the *perturbed estimator* of δ for the i -th small area, defined by

$$\check{\delta}_i = \begin{cases} \bar{\delta}_i & \text{if } \bar{\delta}_i \in \Delta \text{ and } |M_{1i}^{(1)}(\hat{\delta})|^{-1} \leq (1 + \log m)^2 \\ \hat{\delta} & \text{otherwise,} \end{cases} \quad (4.6)$$

and Δ is the set of possible values of the parameter δ under model (2.1). Note that by construction, the MSPE estimator is always nonnegative. Further, LM show that under some regularity conditions, the bias of the estimator $mspe_{\text{LM}}(\hat{\beta}_i)$ is of the order $o(m^{-1})$.

Remark 3: When more than one partial derivatives $M_{1i}^{(j)}(\delta)$ are non-zero, one may use perturbations along all such directions. Thus, an alternative MSPE estimator is given by

$$mspe_{\text{LM:ALT}}(\hat{\beta}_i) = M_{1i}(\check{\delta}_i^\dagger) + M_{2i}(\hat{\delta}), i = 1, \dots, m, \quad (4.7)$$

where

$$\check{\delta}_i^\dagger = \begin{cases} \delta_i^\dagger & \text{if } \delta_i^\dagger \in \Delta \text{ and } |\mathcal{J}|^{-1} \sum_{j \in \mathcal{J}} |M_{1i}^{(j)}(\hat{\delta})|^{-1} \leq (1 + \log m)^2 \\ \hat{\delta} & \text{otherwise,} \end{cases},$$

$\delta_i^\dagger = \hat{\delta} - \sum_{j \in \mathcal{J}} [\hat{B}_i / M_{1i}^{(j)}(\hat{\delta})] e_j / |\mathcal{J}|$, $\mathcal{J} = \{j : 1 \leq j \leq k, M_{1i}^{(j)}(\delta) \neq 0\}$ and for any set A , let $|A|$ denotes its size. The arguments developed in LM readily imply that the new MSPE estimator is also second order correct, under the same set of regularity conditions as in LM. By combining all $|\mathcal{J}|$ directions, the new estimator attains a better finite sample stability.

4.2 Nonnegative estimation of the MSPE when expressions for M_{1i} and M_{2i} are Unavailable

As discussed earlier, except for very few standard models, exact or closed form expressions for the terms $M_{1i}(\cdot)$ and $M_{2i}(\cdot)$ are not available. Here we employ the Bootstrap method of Efron (1979) to develop an approximated version of the estimator $mspe_{\text{LM}}$ that is nonnegative,

second order accurate, and that can be computed without additional analytical work on the part of the user. To that end, first we define a bootstrap based approximation to the function $M_{1i}(\cdot)$ at a given value δ_0 . Let $(y_i^{*l}, \theta_i^{*l})$, $l = 1, \dots, N_0$ be iid random vectors generated using model (2.1) with $\delta = \delta_0$. Then the bootstrap approximation to $M_{1i}(\delta_0)$ is given by

$$M_{1i}^*(\delta_0) = \frac{1}{N_0} \sum_{l=1}^{N_0} \left\{ \xi_i(y_i^{*l}; \delta_0) - h(\theta_i^{*l}) \right\}^2. \quad (4.8)$$

Next we use $M_{1i}^*(\cdot)$ to construct estimators of the partial derivatives of the function $M_{1i}(\cdot)$. To motivate the construction, consider a smooth function $f : \mathbb{R} \rightarrow \mathbb{R}$. Then, for any $a \in \mathbb{R}$,

$$f(a + \epsilon) - f(a - \epsilon) = \{f(a + \epsilon) - f(a)\} - \{f(a - \epsilon) - f(a)\} = 2\epsilon f'(a) + o(\epsilon)$$

as $\epsilon \rightarrow 0$, where $f'(a)$ denotes the derivative of $f(\cdot)$ at a . Hence the scaled difference $(2\epsilon)^{-1}\{f(a + \epsilon) - f(a - \epsilon)\}$ gives an approximation to $f'(a)$ for small values of ϵ . We now employ this fact to define suitable approximations to the first order partial derivatives of $M_{1i}(\cdot)$ at $\hat{\delta}$. Let $\{z_m\}$ be a sequence of positive real numbers converging to zero. Let

$$M_{1i}^{(j)*}(\hat{\delta}) = \frac{1}{2z_m} \left\{ M_{1i}^*(\hat{\delta} + z_m e_j) - M_{1i}^*(\hat{\delta} - z_m e_j) \right\}, \quad (4.9)$$

$j = 1, \dots, k$. Using a similar reasoning, we also define approximations to the second order partial derivatives as

$$M_{1i}^{(j,j)*}(\hat{\delta}) = \frac{1}{[z_m]^2} \left\{ M_{1i}^*(\hat{\delta} + z_m e_j) + M_{1i}^*(\hat{\delta} - z_m e_j) - 2M_{1i}^*(\hat{\delta}) \right\}, \quad 1 \leq j \leq k, \quad (4.10)$$

$$\begin{aligned} M_{1i}^{(j,r)*}(\hat{\delta}) &= \frac{1}{2z_m^2} \left[\left\{ M_{1i}^*(\hat{\delta} + z_m e_{j,r}) + M_{1i}^*(\hat{\delta} - z_m e_{j,r}) - 2M_{1i}^*(\hat{\delta}) \right\} \right. \\ &\quad \left. - z_m^2 \left\{ M_{1i}^{(j,j)*}(\hat{\delta}) + M_{1i}^{(r,r)*}(\hat{\delta}) \right\} \right], \quad 1 \leq j \neq r \leq k, \end{aligned} \quad (4.11)$$

where $e_{j,r} = e_j + e_r$. Theorem 1 in Section 5 shows that under some regularity conditions, $\max_{1 \leq i \leq m} E|M_{1i}^{(j)*}(\hat{\delta}) - M_{1i}^{(j)}(\hat{\delta})| = O(z_m + (z_m)^{-1}N_0^{-\eta/(1+\eta)})$. and $\max_{1 \leq i \leq m} E|M_{1i}^{(j,r)*}(\hat{\delta}) - M_{1i}^{(j,r)}(\hat{\delta})| = O(z_m + (z_m)^{-2}N_0^{-\eta/(1+\eta)})$. for all $1 \leq j, r \leq k$, for some $\eta \in (0, 1]$. Thus, the proposed estimators of the partial derivatives provide accurate approximations for suitable choices of z_m and N_0 .

Next for $l = 1, \dots, N_0$, let $(y_1^{*l}, \dots, y_m^{*l})$ be iid random vectors having joint distribution (2.1) with $\delta = \hat{\delta}$ and let δ^{*l} denote the bootstrap version of $\hat{\delta}$, obtained by replacing

(y_1, \dots, y_m) with $(y_1^{*l}, \dots, y_m^{*l})$. Define the bootstrap estimators of the bias and the variance of $\hat{\delta}$ by

$$b^* = \frac{1}{N_0} \sum_{l=1}^{N_0} \delta^{*l} - \hat{\delta} \quad \text{and} \quad V^* = \left\{ \frac{1}{N_0} \sum_{l=1}^{N_0} \delta^{*l} (\delta^{*l})^T \right\} - \left(\frac{1}{N_0} \sum_{l=1}^{N_0} \delta^{*l} \right) \left(\frac{1}{N_0} \sum_{l=1}^{N_0} \delta^{*l} \right)^T, \quad (4.12)$$

respectively. Theorem 2 in Section 5 below gives conditions for the consistency of b^* and V^* . With this, we now define the *bootstrap based preliminary perturbed estimator* $\bar{\delta}_i^*$ as

$$\bar{\delta}_i^* = \hat{\delta} - B_i^* \{M_{1i}^{(s)^*}(\hat{\delta})\}^{-1} e_s, \quad \text{provided} \quad |M_{1i}^{(s)^*}(\hat{\delta})| \neq 0 \quad \text{for some} \quad s \equiv s_i \in \{1, \dots, k\},$$

where $B_i^* = \sum_{j=1}^k M_{1i}^{(j)*}(\hat{\delta}) b^*(j) + 2^{-1} \sum_{j=1}^k \sum_{r=1}^k M_{1i}^{(j,r)*}(\hat{\delta}) V^*(j, r)$, $x(j)$ denotes the j th component of a vector x and $B(j, r)$ denotes the (j, r) th element of a matrix B . The *bootstrap based perturbed estimator* of δ for the i th small area is now defined as

$$\check{\delta}_i^* = \begin{cases} \bar{\delta}_i^* & \text{if } \bar{\delta}_i^* \in \Delta \text{ and } |M_{1i}^{(s)^*}(\hat{\delta})|^{-1} \leq (1 + \log m)^2 \\ \hat{\delta} & \text{otherwise} \end{cases} \quad (4.13)$$

and the bias corrected estimator of $M_{1i}(\delta)$ is given by $M_{1i}^*(\check{\delta}_i^*)$, $i = 1, \dots, m$.

Next we define the bootstrap estimator of $M_{2i}(\delta)$. Note that $M_{2i}(\delta) = E_{\delta}(\hat{\beta}_i - \tilde{\beta}_i)^2 = E_{\delta}\{\xi_i(y_i, \hat{\delta}) - \xi_i(y_i, \delta)\}^2$. Let δ^{*l} , $l = 1, \dots, N_0$ denote iid bootstrap replicates of $\hat{\delta}$ as above (cf. (4.12)). Then, the parametric bootstrap estimator of $M_{2i}(\delta)$ is now defined as

$$M_{2i}^*(\hat{\delta}) = N_0^{-1} \sum_{l=1}^{N_0} \{\xi_i(y_i^{*l}, \delta^{*l}) - \xi_i(y_i^{*l}, \hat{\delta})\}^2. \quad (4.14)$$

Peffermann and Tiller (2005), Pfeffermann and Glickman (2004) and Butar and Lahiri (2003) also proposed similar parametric bootstrap estimates of $M_{2i}(\cdot)$ for normal errors.

The proposed *bias corrected estimator* of the MSPE $M_i(\delta)$ is defined as

$$mspe_{\text{NEW}}(\hat{\beta}_i) = M_{1i}^*(\check{\delta}_i^*) + M_{2i}^*(\hat{\delta}), \quad (4.15)$$

$i = 1, \dots, m$. In the next section, we show that under some regularity conditions, the proposed estimator has a bias that is of the order $o(m^{-1})$. As a result, the proposed estimator attains the same level of asymptotic bias accuracy as the previously proposed MSPE estimators. Furthermore, as (4.15) does not require explicit expressions for the functions M_{1i} and M_{2i} , the proposed MSPE estimation methodology can be applied to complex or nonstandard models where none of the existing methods are easily applicable.

5 THEORETICAL PROPERTIES

For investigating the theoretical properties of the proposed method, we shall suppose that the random variables $(y_i, \theta_i) : i = 1, \dots, m$ and the various bootstrap variables $(y_i^{*l}, \theta_i^{*l})$'s are defined on a common probability space. We write P_x and E_x to denote the probability and the expectation under a given parameter value $x \in \Delta$. For notational simplicity, we set $P_\delta = P$ and $E_\delta = E$ where δ is the true value of the parameter. Define the functions $a(\cdot)$ and $\Sigma(\cdot)$ by $b(\delta) = a(\delta)/m$ and $V(\delta) = \Sigma(\delta)/m$, where $b(\delta) \equiv E_\delta(\hat{\delta}) - \delta$ and $V(\delta) = \text{var}_\delta(\hat{\delta})$. Note that a, b, Σ and V depend on m . Unless otherwise specified, limits in the order symbols below are taken as $m \rightarrow \infty$. Also, let $E_{\cdot|\hat{\delta}}$ denote the conditional expectation of the bootstrap variables given $\hat{\delta}$. Proofs of the main results are given in the Appendix.

Conditions

- (C.1) (i) δ , the true value of the parameter, is an interior point of Δ .
- (ii) M_{1i} is three times continuously differentiable on Δ and there exists a constant $C_1 \in (0, \infty)$ such that for all $x \in \Delta, j, r, s = 1, \dots, k$ and $i = 1, \dots, m, m \geq 1$, $|M_{1i}^{(j)}(x)| + |M_{1i}^{(j,r)}(x)| + |M_{1i}^{(j,r,s)}(x)| < C_1$.
- (iii) M_{2i} is differentiable on Δ and there exist constants $C_2, C_3, \epsilon_0 \in (0, \infty)$ and $\gamma \in (0, 1]$ and a function $G_1 : \mathbb{R}^k \rightarrow [0, \infty)$ with $EG_1(\hat{\delta}) = O(1)$ such that for all $j = 1, \dots, k; i = 1, \dots, m, m \geq 1$, $|M_{2i}^{(j)}(\delta)| \leq C_2 m^{-1}, |M_{2i}(x)| \leq m^{-1} G_1(x)$ for all $x \in \Delta$ and $m|M_{2i}^{(j)}(x) - M_{2i}^{(j)}(\delta)| \leq C_3 \|x - \delta\|^\gamma$ for all $x \in \mathcal{N}$, where $\mathcal{N} \equiv \{\|x - \delta\| \leq \epsilon_0\}$.
- (C.2) There exist constants $\eta \in (0, 1]$ and $C_4 \equiv C_4(\eta) \in (0, \infty)$ such that $E|h(\theta_i)|^{2+2\eta} < C_4$ for all $i = 1, \dots, m, m \geq 1$.
- (C.3) (i) Let $\rho_m(x; a) = E_x \|\hat{\delta} - x\|^a, x \in \Delta, a \in (0, \infty)$. Suppose that there exists a constant $\eta \in (0, 1]$ such that $E\rho_m(\hat{\delta}; 2 + 2\eta) = O(1)$.
- (ii) The sequences of functions $\{a\} \equiv \{a_m\}$ and $\{\Sigma\} \equiv \{\Sigma_m\}$ are (component-wise) equicontinuous at δ .
- (iii) There exists a continuous function $G_2 : \mathbb{R}^k \rightarrow [0, \infty)$ such that $\|a_m(x)\| + \|\Sigma_m\| \leq G_2(x)$ for all $x \in \Delta$ and $EG_2(\hat{\delta})^2 = O(1)$.

(C.4) There exists a constant $\eta \in (0, 1]$ such that $E\{\sqrt{m}|\hat{\delta} - \delta|\}^{2+2\eta} = O(1)$.

We now briefly comment on the regularity conditions. Condition (C.1) requires the functions M_{1i} and M_{2i} to be smooth, which typically holds under suitable smoothness conditions on the parametric model (2.1). As mentioned earlier, in most applications the function M_{1i} is of the order $O(1)$ while M_{2i} is of the order $O(m^{-1})$ as $m \rightarrow \infty$. Condition (C.1) requires that the partial derivatives of these functions also have the same orders. Conditions (C.2), (C.3)(i), and (C.4) are moment conditions depending on η , whose values will be specified in the statements of the results below. These are used to prove ‘closeness’ of various parametric bootstrap estimates to their conditional expectations. Condition (C.3)(ii) and (iii) are exclusively used to establish consistency of the bootstrap estimators of the bias and the variance estimators of $\hat{\delta}$.

The first result proves consistency of the partial derivative estimates.

Theorem 1: Let Conditions (C.1)(ii) and (C.2) hold and let N_0 be as in (4.8). Then

$$\max_{1 \leq j \leq k} \max_{1 \leq i \leq m} E|M_{1i}^{(j)*}(\hat{\delta}) - M_{1i}^{(j)}(\hat{\delta})| = O(z_m + [z_m]^{-1} N_0^{-\frac{\eta}{1+\eta}}), \quad (5.1)$$

$$\max_{1 \leq j, r \leq k} \max_{1 \leq i \leq m} E|M_{1i}^{(j,r)*}(\hat{\delta}) - M_{1i}^{(j,r)}(\hat{\delta})| = O(z_m + [z_m]^{-2} N_0^{-\frac{\eta}{1+\eta}}). \quad (5.2)$$

Note that the right sides of (5.1) and (5.2) go to zero for any $z_m \rightarrow 0$, $N_0 \rightarrow \infty$ such that $z_m^2 N_0^{\eta/(1+\eta)} \rightarrow \infty$. Here z_m acts as a smoothing parameter that controls the bias parts of the proposed estimates. For a smaller value of z_m , a larger value of the resample size N_0 has to be chosen accordingly to attain a desired accuracy level. Also, note that the value of N_0 required for estimating the second order partial derivatives must grow at a faster rate than the case of the first order partial derivatives to attain the same level of accuracy.

The next result considers accuracy of the bootstrap bias and variance estimators of $\hat{\delta}$.

Theorem 2: Let Condition (C.3) hold and let N_0 be as in (4.8). Then,

$$E\|b^* - \hat{b}\|^2 = O(N_0^{-1}) \quad \text{and} \quad E\|V^* - \hat{V}\| = O(N_0^{-\frac{\eta}{1+\eta}}).$$

Under the conditions of Theorem 2, the bootstrap bias estimator is $N_0^{1/2}$ -consistent. The variance estimator can also attain the same rate, provided $\eta = 1$. Note that unlike Theorem

1, the estimators of the bias and the variance matrix of $\hat{\delta}$ do not involve a smoothing parameter like z_m .

The next result shows that under suitable conditions, the proposed estimator of the $\text{MSPE}(\hat{\beta}_i)$ is second order bias corrected.

Theorem 3: Suppose that conditions (C.1)-(C.4) hold and that $\eta = 1$ in both (C.2) and (C.3). Suppose that for each $i = 1, \dots, m$, there exists $s = s(i) \in \{1, \dots, k\}$ such that

$$|M_{1i}^{(s)}(\delta)| > C_0 \quad (5.3)$$

for all $m \geq 1$, where $C_0 \in (0, \infty)$. Let $z_m = m^{-5/4}$ and $N_0 \geq m^a$ for some $a > 9/2$. Then the proposed mspe estimator is second order bias accurate, i.e.,

$$\max_{1 \leq i \leq m} \left| E \left[\text{mspe}_{\text{NEW}}(\hat{\beta}_i) \right] - M_i(\delta) \right| = o(m^{-1}). \quad (5.4)$$

Theorem 3 shows that the proposed MSPE estimator achieves the same second order bias accuracy as the earlier methods proposed in the literature. Thus, under the given regularity conditions, the additional randomness induced by several resampling steps has a negligible effect on the bias of the new estimator. Since it also does not require the knowledge of the functions $M_{1i}(\cdot)$, $M_{2i}(\cdot)$, of their partial derivatives, and of the bias and variance of the estimator $\hat{\delta}$, the proposed method can be applied to any model of the form (2.1), where the other methods are not readily applicable. The price paid for this omnibus solution is that it is computationally intensive.

6 PRACTICAL IMPLEMENTATION AND NUMERICAL FINDINGS

6.1 Finite sample considerations

In this section, we provide some guidelines for implementing the proposed MSPE estimation methodology in finite sample applications. Supposing, for the time being, that an expression for the BP is known, computation of different parts of the estimator mspe_{NEW} involves generating (parametric) bootstrap samples from the joint distribution of (y_i, θ_i) for $i = 1, \dots, m$ (cf. (2.1)) at various values of the parameter δ . For the bootstrap bias and variance

estimators \hat{b} and \hat{V} and the term $M_{2i}^*(\hat{\delta})$, we suggest using a resample size (drawn from (2.1) with $\delta = \hat{\delta}$) in the 100s (e.g., in the range 500-1000). This is known to be adequate for Monte-Carlo evaluation of bootstrap estimators of variance-type functionals (cf. Efron and Tibshirani (1993)). Next consider numerical evaluation of the first term of $m\text{spe}_{\text{NEW}}$, i.e., of $M_{1i}^*(\check{\delta}_i^*)$. This requires us to approximate the partial derivatives of $M_{1i}(\cdot)$ which, in turn, involve the smoothing parameter z_m . For all computations done in this section, we set $z_m = m^{-5/4}$ as in Theorem 3, although other choices of $z_m \ll m^{-1}$ may be used. For the numerical approximation of the partial derivatives, the resample sizes must be larger in order to compensate for the effect of the smoothing - the smaller the choice of the smoothing parameter z_m , the larger the choice of N_0 will have to be. For m of moderate size (e.g., $m \in (10, 80)$) and z_m as above, we have found resamples of size N_0 in the range 2000-10,000 adequate for computing the first order partial derivatives $M_{1i}^{*(j)}$ and resamples of size $N_0 \approx 10,000$ for the second order ones $M_{1i}^{*(j,r)}$. Finally, in the case that an exact expression for the EBP is not available and it is approximated numerically using (3.6) or (3.7), the resample size J may be chosen in the 100s (e.g., 300-1000) in the discrete case while it must be of a higher order (e.g., 1000+) in the continuous case. Approximations given by the above choices of the resample sizes are generally very good. In the next section, we report the results of a simulation study and the associated computing time for three specific examples where we follow the finite sample guidelines given above. For an illustration, Table 1 below gives the resulting approximations for the EBP both in the discrete and the continuous cases which appear to be in good agreement with the true values.

6.2 *Simulation results*

In this section, we check the performance of the MSPE estimators (4.5) and (4.15) for Models I-III described below, and compare them with the Datta-Lahiri (2000) (hereafter, referred to as DL) version of the PR method and the jackknife method of JLW, as described in Rao (2003). DL extended the PR method when the model parameters are estimated using MLEs. We used MLEs of the model parameters for Models I and II, and used estimating equations for Model III. Normal kernel was used for the kernel based EBP's. We use the following

notations for different methods of MSPE estimation: JK for jackknife, LM1 for (4.5) and LM2 for (4.15).

Model I: Normal-Normal. This is a continuous data model, where both F_1 and F_2 are normal; The model structure is specified by (2.2) with $\lambda = 0$. In this setting, all four methods of bias correction are applicable. Although in this case a closed form expression for the BP exists, to gain some insight into the performance of the suggested approximations, we use (3.6) to find the BP for the LM2 method. For the other three methods, the available closed form expressions are used. We choose F_2 to be normal with mean 0 and variance unity, and F_1 to be normal with mean 0 and variance $s_i, i = 1, \dots, m$. with $m=15$. The 15 areas are divided into three groups of five, with equal numbers of areas and equal values of s_i . The three different values of s_i used are (.7, .5, .3). The set-up is similar to the one considered by Datta *et al.* (2005).

Model II: Binomial-‘Logit-Normal’. This is a binary data model where we suppose that F_1 is binomial and F_2 is logit-normal. In particular, the logit of the success probability of F_1 is normally distributed with mean zero and variance unity (cf. (2.4) with $\lambda = 0$). In this setting, only JK and LM2 methods of MSPE estimation are applicable. The binomial population has 8 areas, of respective sizes $n_i=36, 20, 19, 16, 17, 11, 5$ and 6, based on the number of patients receiving a particular treatment from different clinics (Booth and Hobert, 1998). To generate the i th binomial population, we first generate the success probability

$$p_i = \frac{\exp(\mu + v_i)}{1 + \exp(\mu + v_i)} \quad (6.1)$$

where v_i is a standard normal variate, $i = 1, \dots, 8$ and $\mu = 0$. In this case the BP is not available in a closed form. We first find the maximum likelihood estimates of the model parameters using Slud (2000). Then the BP is calculated using Gauss-Hermite quadrature with 15 points for the JK method and (3.7) for the LM2 method.

Model III: Normal-Lognormal. This is a continuous data non-conjugate model, where F_1 is normal and F_2 is lognormal. You and Rao (2002) considered this model for estimating the Canadian census undercoverage and called this as ‘unmatched sampling and linking model’. Here, neither the PR/DL nor the JK methods are applicable in a straightforward

way. We took $m = 15$ and generated θ_i 's ($i = 1, \dots, m$) from a lognormal distribution. We took two covariates, besides the intercept, one was generated from $N(0, .5)$ and the other was generated from Uniform $(.5, 1)$. We set $\delta = (\lambda^T, \sigma_v^2)^T = (0, 0.5, -1.5, 0.5)^T$. Then given θ_i 's, y_i 's were generated as in Model I. Instead of using ML estimate, we used unbiased estimating equation approach for estimating the model parameters (cf. Ghosh and Maiti, 2004). Since the BP does not have any closed form expression, we used the kernel based estimator (3.6) for estimating the BP and consequently, of the four, here LM2 is the only method available for estimating the MSPE. Also, note that in this case, one can obtain the *perturbed estimator* of δ either by (4.15) or by the (estimated version of the) method described in Remark 3. Both methods gave very similar results. The MSPE estimator in Remark 3 (with estimated partial derivatives, etc.) gave slightly low CV than the estimator in (4.15); see Table 2.

In implementing LM2, we used 1000 bootstrap samples for finding the bias and variances estimates and 10000 bootstrap samples for all other approximations. All simulation results were based on R=1000 replication. The approximate computation time for each model is at most 48 hours on a UNIX workstation equipped with 4000MHz 64-bit CPU and FORTRAN 77 compiler. In any real application, user needs to run the code only once, meaning minimal computational time (less than 3 minutes) with data sets of a similar size.

To study the performance of the EBP $\hat{\theta}_i$ of the small area parameter θ_i , we use the following two empirical measures.

$$\text{Absolute relative bias} \quad T_1 = \frac{1}{R} \sum_{r=1}^R \left| \frac{\hat{\theta}_i^{(r)} - \theta_i^{(r)}}{\theta_i^{(r)}} \right|. \quad (6.2)$$

$$\text{Empirical MSPE} \quad T_2 = \frac{1}{R} \sum_{r=1}^R (\hat{\theta}_i^{(r)} - \theta_i^{(r)})^2. \quad (6.3)$$

The body of all the tables gives averages over all the small areas where the “average” is measured in terms of the median (given in the first column for each model) or the mean (in the second column).

Table 1. *Absolute relative bias (T_1) and empirical MSPE (T_2) for the EBP. Results using the kernel based approximations (3.6) and (3.7) are reported within the parentheses.*

	Model I		Model II		Model III	
Measures	Median	Mean	Median	Mean	Median	Mean
T_1	2.318	4.171	0.223	0.243	—	—
	(2.156)	(4.122)	(0.224)	(0.243)	(0.926)	(0.923)
T_2	0.376	0.366	0.0107	0.0131	—	—
	(0.378)	(0.373)	(0.0107)	(0.0131)	(0.269)	(0.292)

There is a good agreement between the actual values and the approximations for the EBP given in equations (3.6) and (3.7). For the binary data, this agreement is particularly remarkable. This is because for the same value of the resample size J , the approximation in the discrete case is more accurate (having a faster rate of convergence). In the case of the binary data, the “actual” values are found by numerical integration. The simulation result shows that both the numerical integration based approximation and the “kernel” method based approximation (3.6) behave similarly. However, kernel method seems more automated than numerical integration as it does not require additional programming for a different continuous data model.

Table 2 reports the following empirical measures of relative bias and coefficient of variation, quantifying the performances of different MSPE estimation methods:

$$\text{Relative bias } T_3 = [E\{M\hat{S}PE(\hat{\theta}_i)\} - T_2]/T_2 \quad (6.4)$$

$$\text{Coefficient of variation } T_4 = \left[E\{M\hat{S}PE(\hat{\theta}_i) - T_2\}^2 \right]^{\frac{1}{2}} / T_2. \quad (6.5)$$

Here $E\{M\hat{S}PE(\hat{\theta}_i)\}$ and $E\{M\hat{S}PE(\hat{\theta}_i) - T_2\}^2$ are estimated empirically by averaging the replicates of $M\hat{S}PE(\hat{\theta}_i)$ and $\{M\hat{S}PE(\hat{\theta}_i) - T_2\}^2$, respectively.

Table 2. *Relative biases (T_3) and coefficient of variations (T_4) for the bias corrected estimators of the MSPE. Entries within parentheses represent LM1 and LM2 estimates based on Remark 3 modification.*

Method	Measures	Model I		Model II		Model III	
		Median	Mean	Median	Mean	Median	Mean
PR/DL	T_3	-0.016	-0.004	—	—	—	—
	T_4	0.159	0.150	—	—	—	—
JK	T_3	0.068	0.095	-0.088	-0.026	—	—
	T_4	0.504	0.635	0.686	0.758	—	—
LM1	T_3	-0.015	-0.018	—	—	—	—
		(-0.000)	(0.050)	—	—	—	—
	T_4	0.158	0.151	—	—	—	—
		(0.153)	(0.149)	—	—	—	—
LM2	T_3	-0.013	-0.028	-0.108	-0.083	0.116	0.041
		(-0.019)	(-0.024)	(-0.087)	(-0.083)	(0.115)	(0.044)
	T_4	0.229	0.224	0.172	0.164	0.319	0.368
		(0.225)	(0.218)	(0.170)	(0.156)	(0.310)	(0.298)

For Model I, all the methods perform well in terms of minimizing relative bias. However, in terms of the coefficient of variation, there is a difference in the performance of the four methods. The PR/DL and LM1 methods turn out to be the best, followed by the LM2 method. The small increase in the variation of the LM2 method over the LM1 method is expected, as the randomness in the various approximation steps in its construction adds to the total variability of the bias corrected MSPE estimator. However, the highest variation for this model is observed for the JK method, where the variation more than *double* compared to the LM2 method and it is more than *three* times compared to the LM1 and PR/DL methods.

As mentioned earlier, for Model II, only the LM2 and the JK methods are applicable. In this case, the LM2 tends to have higher relative bias. However, in terms of the coefficient of variation, which gives the *combined* effects of the bias and the variance of the MSPE

estimators, the LM2 method again beats the JK method by a relative magnitude of 300% to 400% or more. To gain further insight into the bias properties of the two methods, we repeated the simulation study with $m = 16$ areas (instead of the $m = 8$ areas considered earlier) under Model II. For this higher value of m , we found that the relative bias for the LM2 method dropped to -.038 and -.024 for the median and the mean, respectively. The eight additional small area sizes were 37, 32, 19, 17, 12, 10, 9 and 7. In comparison, the relative bias for the JK method under $m = 16$ were -.025 and -.026 for the median and mean respectively. The coefficient of variations for the two methods continued to show a similar pattern as in the $m = 8$ case. Thus for both models, the estimators produced by the JK method have inferior performance in terms of the coefficient of variation.

For Model III, the PR/DL method is not applicable and the existing literature does not show how to apply the JK method. This is a somewhat unusual set up of simulation within the existing SAE literature. It may be interesting to know that, if some one naively used $M_1(\hat{\delta})$ with formula (4.8), the median relative bias would be -.227 and the mean, -.260. This indicates severe under-estimation which is expected. In comparison, LM2 produces satisfactory results for both the relative bias and the coefficient of variation.

7 DISCUSSION

In this paper, we consider a new method of bias correction for the “simple” estimator of the MSPE of a possibly nonlinear function of the small area means $h(\theta_i), i = 1, \dots, m$. The proposed method may be contrasted with the existing methods, which require explicit analytical expressions for bias correction. The popular linearization method of bias correction proposed by PR can not be easily extended to nonlinear h and non-normal models. Further the PR approach is sensitive to the method of estimating model parameters in the sense that additional analytical work may be needed for each new estimation method. In the cases where exact analytical expressions are available, the simulation results indicate that the PR and the LM methods (are comparable and) have the best overall performance (in terms of MSEs) while the proposed method (LM2) fares reasonably well against these. In particular, LM is preferable to LM2 in such situations. As for comparison with the JK method in this

case, the LM2 method performs much better than the JK method in finite samples in terms of the co-efficient of variation.

In the more complicated examples, where exact analytical expressions for the MSPE are not available, the LM and the PR methods are not applicable, but the LM2 method and the JK method (with some suitable adaptation) are. In this case, the LM2 method seems to have a superior performance compared to the JK method in terms of overall accuracy. Further, because of the inherent limitations of the Jackknife method for estimating the variance of a non-smooth estimator of the model parameters (e.g., the sample median), the JK method may produce an inconsistent estimator of the MSPE (more precisely, of the variance type term M_{2i}), while the bootstrap based LM1 and LM2 methods would still work (cf. Ghosh et al. (1984)). From this point of view, the proposed method of MSPE estimation has a wider range of validity than the JK method.

In this paper, we also prove that the proposed estimator of the MSPE attains the same level of asymptotic accuracy as the existing methods in correcting the bias of the simple MSPE estimator. We also report the results of a small simulation study and provide some guidelines for implementing the methodology in practice. In summary, the proposed method allows a user to routinely derive second order accurate, nonnegative estimates of the MSPE in small area estimation problems, without requiring any analytical work on the part of the user.

ACKNOWLEDGEMENT

The authors thank three referees for their constructive criticism that led to a vast improvement of an earlier draft of the paper. The authors also thank Douglas Williams for some helpful discussions and for facilitating the project.

REFERENCES

- BOOTH, J.G., & HOBERT, J.P. (1998). Standard errors of prediction in generalized linear mixed models. *J. Am. Statist. Assoc.* **83**, 28-36.
- BUTAR, F.B., & LAHIRI, P. (2003). On measures of uncertainty of empirical Bayes small area estimators. *J. Statist. Plan. Infer.* **112**, 63-76.

- DATTA, G.S. & LAHIRI, P. (2000). A unified measure of uncertainty of estimated best linear unbiased predictors in small area estimation problems. *Statist. Sinica*, **10**, 623-27.
- DATTA, G.S., RAO, J.N.K., & SMITH, D.D. (2005). On measuring the variability of small area estimators under a basic area level model. *Biometrika* **92**, 183-96.
- EFRON, B. (1978). Regression and ANOVA with zero-one data: Measures of residual variation. *J. Am. Statist. Assoc.* **73**, 113-21.
- EFRON, B. (1979). Bootstrap methods: Another look at the jackknife. *Ann. Statist.* **7**, 1-26.
- EFRON, B. (1986). Double exponential families and their use in generalized linear regression. *J. Am. Statist. Assoc.* **81**, 709-21.
- EFRON, B. & TIBSHIRANI, R. (1993). *An introduction to the bootstrap*. Chapman & Hall Ltd., New York.
- FAY, R. E. & HERRIOT, R. A. (1979). Estimates of income for small places: An application of James-Stein procedures to census data. *J. Am. Statist. Assoc.* **74**, 269-77.
- GHOSH, M., & MAITI, T. (2004). Small-area estimation based on natural exponential family quadratic variance function models and survey weights. *Biometrika* **91**, 95-112.
- GHOSH, M., PARR, W. C., SINGH, K., & BABU, G.J. (1984). A note on bootstrapping the sample median. *Ann. Statist.* **12**, 1130-35.
- HÄRDLE, W. (1991). *Smoothing Techniques: With implementation in S*. Springer, New York, NY.
- JIANG, J., LAHIRI, P., & WAN, S-M. (2002). A unified jackknife theory for empirical best prediction with M-estimation. *Ann. Statist.* **30**, 1782-810.
- LAHIRI, S.N. & MAITI, T. (2003). Nonnegative Mean Squared Error prediction. *Preprint*.
Posted at <http://arxiv.org/abs/math.ST/0604075>.

- LAHIRI, S.N., MAITI, T., KATZOFF, M., & PARSONS, V. (2006). Resampling based empirical prediction: An application to small area estimation. Posted at <http://arxiv.org/abs/math.ST/0604513>.
- MCCULLOCH, C.E., & SEARLE, S.R. (2001). *Generalized, Linear, and Mixed Models* New York: Wiley.
- PFEFFERMANN, D. & GLICKMAN, H. (2004). Mean squared error approximation in small area estimation by use of parametric and nonparametric bootstrap. *Proc. Sec. Survey Res. Meth., Am. Statist. Assoc.*
- PEFFEREMANN, D., & TILLER, R.B. (2005). Bootstrap approximation to prediction MSE for state-space models with estimated parameters. *J. Time Ser. Analysis*, **26**, 893-16.
- PRASAD, N.G.N. & RAO, J.N.K. (1990). The estimation of the mean squared error of small area estimators. *J. Am. Statist. Assoc.* **68**, 67-72.
- RAO, J.N.K. (2003). *Small Area Estimation*. Wiley, New York.
- SLUD, E.V. (2000). Comparison of aggregate versus unit-level models for small area estimation. *Proc. Sec. Survey Res. Meth., Am. Statist. Assoc.*
- SLUD, E.V., & MAITI, T. (2006). MSE estimation in transformed Fay-Herriot models. *J. R. Statist. Soc. B* **68**, 239-57.
- YOU, Y. & RAO, J.N.K. (2002). Small area estimation using unmatched sampling and linking models. *Can. J. Statist.* **30**, 3-15.

APPENDIX A: PROOFS

Let $\mathcal{N} = \{1, 2, \dots\}$. In the proofs, we suppress dependence of various quantities on m unless there is a chance of confusion and write $C, C(\cdot)$ to denote generic positive constants that depend on their arguments (if any), but not on $i \in \{1, \dots, m\}$ or m .

Lemma 1: Let X_1, \dots, X_n ($n \geq 1$) be a collection of iid random variables with $E|X_1|^{1+\eta} < \infty$ for some $\eta \in (0, 1]$. Let $\mu = EX_1$, $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$ and $\rho = (E|X_1|^{1+\eta})^{\frac{1}{1+\eta}}$. Then

$$E|\bar{X}_n - \mu| \leq 3\rho n^{\eta/(1+\eta)} \quad \text{for all } n \geq 1. \quad (\text{A.1})$$

Proof: If $\rho = 0$, then (A.1) holds trivially. Hence, suppose that $\rho > 0$. With $c_n = \rho n^{1/(1+\eta)}$, let $X_{1i} = X_i I(|X_i| \leq c_n)$, $X_{2i} = X_i - X_{1i}$, $1 \leq i \leq n$, and $\bar{W}_{kn} = n^{-1} \sum_{i=1}^n (X_{ki} - EX_{ki})$, $k = 1, 2$. Then, $E|\bar{X}_n - \mu| \leq E|\bar{W}_{1n}| + E|\bar{W}_{2n}| \leq (n^{-1}E|X_{11}|^2)^{1/2} + 2E|X_{21}| \leq (n^{-1}c_n^{1-\eta}E|X_1|^{1+\eta})^{1/2} + 2E|X_1|^{1+\eta}c_n^{-\eta} \leq 3\rho n^{\eta/(1+\eta)}$.

Lemma 2: For random vectors X and Y on a common probability space with $E|g(Y)|^\alpha < \infty$ for some $g : \mathbb{R}^s \rightarrow \mathbb{R}$ and $\alpha \in [1, \infty)$, $E|E\{g(Y)|X\} - g(Y)|^\alpha \leq 2^\alpha E|g(Y)|^\alpha$.

Proof: Follows from Hölder's and conditional Jensen's inequalities.

Proof of Theorem 1: (i) By (C.1)(ii) and Taylor's expansion, for some $u_{1i}, u_{2i} \in [-1, 1]$,

$$\begin{aligned} & E\left|\left\{M_{1i}(\hat{\delta} + z_m e_j) - M_{1i}(\hat{\delta} - z_m e_j)\right\} - 2z_m M_{1i}^{(j)}(\hat{\delta})\right| \\ &= 2^{-1}[z_m]^2 E\left|M_{1i}^{(j,j)}(\hat{\delta} + u_{1i} z_m e_j) - M_{1i}^{(j,j)}(\hat{\delta} + u_{2i} z_m e_j)\right| \leq C_1[z_m]^2 \end{aligned} \quad (\text{A.2})$$

for all $i = 1, \dots, m$, $m \geq 1$. Since $E_{\delta_0} M_{1i}^*(\delta_0) = M_{1i}(\delta_0)$ for all $\delta_0 \in \Delta$, by Lemmas 1 and 2,

$$\begin{aligned} & E\left|M_{1i}^*(\hat{\delta}_1) - M_{1i}(\hat{\delta}_1)\right| \leq 3E\{E_{\hat{\delta}_1} |\xi_i(y_i^{*1}) - h(\theta_i^{*1})|^{2+2\eta} N_0^{-\eta/(1+\eta)}\} \\ & \leq C(\eta)E\{E_{\hat{\delta}_1} |h(\theta_i^{*1})|^{2+2\eta} N_0^{-\eta/(1+\eta)}\} \leq C(\eta)N_0^{-\eta/(1+\eta)}, \end{aligned} \quad (\text{A.3})$$

where $\hat{\delta}_1 \equiv \hat{\delta} + z_m e_j$. Using (A.3) and similar arguments for $M_{1i}^*(\hat{\delta} - z_m e_j)$, we get

$$E|M_{1i}^{(j)*}(\hat{\delta}) - (2z_m)^{-1}\{M_{1i}(\hat{\delta} + z_m e_j) - M_{1i}(\hat{\delta} - z_m e_j)\}| \leq (2z_m)^{-1}\{C(\eta)N_0^{-\eta/(1+\eta)}\} \quad (\text{A.4})$$

uniformly in $i = 1, \dots, m$, $m \geq 1$. Part (i) of the theorem now follows from (A.2)-(A.4).

Next consider (ii). By arguments similar to (A.2), $E|\{M_{1i}(\hat{\delta} + z_m e_j) + M_{1i}(\hat{\delta} - z_m e_j) - 2M_{1i}(\hat{\delta})\} - (z_m)^2 M_{1i}^{(j,j)}(\hat{\delta})| \leq C \cdot (z_m)^3$ uniformly in $i = 1, \dots, m$, $m \geq 1$. Also, using Lemma 1, the linearity of $M_{1i}^{(j,j)*}(\hat{\delta})$ in $M_{1i}^{(j)*}(\cdot)$ and arguments similar to (A.3), one can show that

$$E|M_{1i}^{(j,j)*}(\hat{\delta}) - (z_m)^{-2}\{M_{1i}(\hat{\delta} + z_m e_j) + M_{1i}(\hat{\delta} - z_m e_j) - 2z_m M_{1i}^{(j)}(\hat{\delta})\}| \leq C(\eta)(z_m)^{-2}N_0^{-\eta/(1+\eta)} \quad (\text{A.5})$$

uniformly in $i = 1, \dots, m$, $m \geq 1$. Hence, part (ii) holds for all $j, r \in \{1, \dots, k\}$ with $j = r$. Next fix $1 \leq j \neq r \leq k$. Define $M_{1i}^{(j,r)\dagger}(\hat{\delta}) = (2[z_m]^2)^{-1}[\{M_{1i}(\hat{\delta} + z_m e_{j,r}) + M_{1i}(\hat{\delta} - z_m e_{j,r}) - 2M_{1i}(\hat{\delta})\} - [z_m]^2\{M_{1i}^{(j,j)}(\hat{\delta}) + M_{1i}^{(r,r)}(\hat{\delta})\}]$. By Taylor's expansion

$$E|M_{1i}^{(j,r)\dagger}(\hat{\delta}) - M_{1i}^{(j,r)}(\hat{\delta})|(2[z_m]^2) \leq C[z_m]^3. \quad (\text{A.6})$$

Now using (A.6) and arguments similar to (A.5), one can complete the proof of (ii).

Proof of Theorem 2: Note that $E_{|\hat{\delta}}(\delta^{*1}) - \hat{\delta} = b(\hat{\delta}) = \hat{b}$ and $E_{|\hat{\delta}}(V^{*1}) = \hat{V}$. Hence, for any $j \in \{1, \dots, k\}$, $E|b^*(j) - \hat{b}(j)|^2 \leq N_0^{-1}E\{E_{|\hat{\delta}}(\delta^{*l}(j) - \hat{\delta}(j))^2\} \leq 2N_0^{-1}E\{\rho_m(\hat{\delta}; 2) + \|\hat{\delta}\|^2\} = O(N_0^{-1})$. Similarly, by Lemma 1, $E|V^*(j, r) - \hat{V}(j, r)|$ is bounded above by

$$\begin{aligned} & E\left[E_{|\hat{\delta}}\left|N_0^{-1}\sum_{l=1}^{N_0}\delta^{*l}(j)\delta^{*l}(r) - \hat{\delta}(j)\hat{\delta}(r)\right|\right. \\ & \quad \left. + \{E_{|\hat{\delta}}|\bar{d}_j^* - \hat{\delta}(j)|^2\}^{1/2}\{E_{|\hat{\delta}}(\bar{d}_r^*)^2\}^{1/2} + |\hat{\delta}(j)|(E_{|\hat{\delta}}|\bar{d}_r^* - \hat{\delta}(r)|^2)^{1/2}\right] \\ & \leq C(\eta)\left[E\{\rho_m(\hat{\delta}; 2 + 2\eta) + \rho_m(\delta; 2 + 2\eta)\}N_0^{-\frac{\eta}{1+\eta}} + E\{\rho_m(\hat{\delta}; 2) + \rho_m(\delta; 2)\}N_0^{-1/2}\right], \end{aligned}$$

for any $j, r \in \{1, \dots, k\}$, where $\bar{d}_j^* = N_0^{-1}\sum_{l=1}^{N_0}\delta^{*l}(j)$. Theorem 2 follows from these bounds.

Lemma 3 : Suppose that condition (C.3) holds. Then, for any $\gamma \in (0, \eta)$, $E\|\hat{b} - b\|^{1+\gamma} = o(m^{-(1+\gamma)})$ and $E\|\hat{V} - V\|^{1+\gamma} = o(m^{-(1+\gamma)})$.

Proof: Fix $\gamma \in (0, \eta)$. Note that $mE\|\hat{\delta} - \delta\|^2 \leq C[\|a_m(\delta)\|^2 + \|\Sigma_m(\delta)\|^2] \leq CG_2(\delta) < \infty$. Hence, $\hat{\delta} \rightarrow \delta$ in mean square and therefore, by the equicontinuity condition, $\|a_m(\hat{\delta}) - a_m(\delta)\|$ and $\|\Sigma_m(\hat{\delta}) - \Sigma_m(\delta)\|$ both converge to zero in probability under δ . Further, the sequence $\{G_2(\hat{\delta})^{1+\gamma}\}$ is uniformly integrable. Hence, by the (extended) Dominated Convergence Theorem, $[E\|a_m(\hat{\delta}) - a_m(\delta)\|^{1+\gamma} + E\|\Sigma_m(\hat{\delta}) - \Sigma_m(\delta)\|^{1+\gamma}] \rightarrow 0$ as $m \rightarrow \infty$, proving the lemma.

Proof of Theorem 3: First we show that

$$\max_{1 \leq i \leq m} E|M_{1i}^*(\check{\delta}_i^*) - M_{1i}(\check{\delta}_i)| + \max_{1 \leq i \leq m} E|M_{2i}^*(\hat{\delta}) - M_{2i}(\hat{\delta})| = o(m^{-1}). \quad (\text{A.7})$$

Consider the first term on the left side. By arguments similar to (A.5), $\max_{1 \leq i \leq m} E|M_{1i}^*(\check{\delta}_i^*) - M_{1i}(\check{\delta}_i^*)| \leq C(\eta)N_0^{-\eta/(1+\eta)}$. Next, write $A_i^* = \{\check{\delta}_i^* \in \Delta\} \cap \{|M_{1i}^{(s)*}(\hat{\delta})|^{-1} \leq (1 + \log m)^2\}$ and

$A_i = \{\bar{\delta}_i \in \Delta\} \cap \{|M_{1i}^{(s)}(\hat{\delta})|^{-1} \leq (1 + \log m)^2\}$, $1 \leq i \leq m$, $m \geq 1$. Then, using (4.13), it can be shown that

$$\begin{aligned} E|M_{1i}(\check{\delta}_i^*) - M_{1i}(\check{\delta}_i)| &\leq E|M_{1i}(\bar{\delta}_i^*) - M_{1i}(\bar{\delta}_i)|I(A_i^* \cap A_i) + E[M_{1i}(\hat{\delta})\{I(A_i^c) + I([A_i^*]^c)\}] \\ &\quad + EM_{1i}(\bar{\delta}_i)I([A_i^*]^c \cap A_i) + EM_{1i}(\bar{\delta}_i^*)I(A_i^* \cap A_i^c) \\ &\equiv R_{1i} + R_{2i} + R_{3i} + R_{4i}, \quad \text{say.} \end{aligned} \quad (\text{A.8})$$

By (C.1), (C.2) (with $\eta = 1$), (C.3) and arguments similar to the proof of Theorem 1, one gets $\max_{1 \leq i \leq m} E|M_{1i}^{(j)*}(\hat{\delta}) - M_{1i}^{(j)}(\hat{\delta})|^2 = O([z_m]^2 + [z_m]^{-2}N_0^{-1})$, $\max_{1 \leq i \leq m} E|M_{1i}^{(j,j)*}(\hat{\delta}) - M_{1i}^{(j,j)}(\hat{\delta})|^2 = O([z_m]^2 + [z_m]^{-4}N_0^{-1})$, and $E\|b\|^2 + E\|V\|^2 = O(m^{-2})$. Now using the above bounds, it can be shown (cf. (A.17), Lahiri et al. (2006)) that

$$\max_{1 \leq i \leq m} R_{1i} \leq C_1 \max_{1 \leq i \leq m} E\|\bar{\delta}_i^* - \bar{\delta}_i\|I(A_i^* \cap A_i) = o(m^{-1}). \quad (\text{A.9})$$

Since $|M_{1i}^{(s)}(\delta)| > C_0$, there exist $\epsilon_1, \epsilon_2 \in (0, \infty)$ such that $|M_{1i}^{(s)}(x)| > \epsilon_1$ for all $x \in \Delta$ with $\|x - \delta\| \leq \epsilon_2$. Hence, by (C.1), there exists a $C = C(\epsilon_1) \in (0, \infty)$ such that on the set $\{\|\hat{\delta} - \delta\| \leq \epsilon_2\}$, $\|\bar{\delta}_i - \delta\| \leq C[\|\hat{b}\| + \|\hat{V}\|]$ for all $i = 1, \dots, m$, $m \geq 1$. Hence, for any $\epsilon > 0$, by (C.1) and (C.4), (cf. (A.18)-(A.19), Lahiri, et al. (2006))

$$\max_{1 \leq i \leq m} P(\|\bar{\delta}_i - \delta\| > \epsilon) \leq P(\|\hat{\delta} - \delta\| > \epsilon_2) + P(C[\|\hat{b}\| + \|\hat{V}\|] > \epsilon) = O(m^{-(1+\eta)}), \quad (\text{A.10})$$

$$\max_{1 \leq i \leq m} P(|M_{1i}^{(s)}(\hat{\delta})| \leq (1 + \log m)^{-2}) \leq 2P(\|\hat{\delta} - \delta\| > C(\epsilon_1, \epsilon_2)) = O(m^{-(1+\eta)}). \quad (\text{A.11})$$

Hence, it follows that

$$\max_{1 \leq i \leq m} P(A_i^c) = O(m^{-(1+\eta)}). \quad (\text{A.12})$$

We now obtain a similar bound on $P([A_i^*]^c)$. Since δ is an interior point of Δ , there exists a $\epsilon_3 \in (0, \infty)$ such that $\{x : \|x - \delta\| \leq \epsilon_3\} \subset \Delta$. Let $A_{1i}^* = \{\bar{\delta}_i^* \in \Delta\}$ and $A_{2i}^* = \{|M_{1i}^{(s)*}(\hat{\delta})|^{-1} \leq (1 + \log m)^2\}$. By (5.3), and (A.7)-(A.12), uniformly over $i = 1, \dots, m$,

$$\begin{aligned} P([A_i^*]^c) &\leq P(A_{1i}^{*c} \cap A_{2i}^* \cap A_i) + P(A_i^c) + P(A_{2i}^{*c}) \\ &\leq P(\|\bar{\delta}_i - \delta\| > \epsilon_3/2) + 2\epsilon_3^{-1}E\|\bar{\delta}_i^* - \bar{\delta}_i\|I(A_{2i}^* \cap A_i) + P(A_i^c) \\ &\quad + \left[P\left(|M_{1i}^{(s)*}(\hat{\delta}) - M_{1i}^{(s)}(\hat{\delta})| > \frac{C_0}{2} - \frac{1}{(1 + \log m)^2}\right) + P\left(|M_{1i}^{(s)}(\hat{\delta}) - M_{1i}^{(s)}(\delta)| > \frac{C_0}{2}\right) \right] \\ &= O\left(m^{-(1+\eta)} + (\log m)^4[z_m + (z_m N_0^{1/2})^{-1}]\right). \end{aligned} \quad (\text{A.13})$$

Now using (A.12), (A.13) and condition (C.1), with $a_i^2 \equiv P(A_i^c) + P([A_i^*]^c)$, we have

$$R_{2i} \leq C\{a_i^2 + C_1 a_i (E\|\hat{\delta} - \delta\|^2)^{1/2}\} = o(m^{-1}), \quad (\text{A.14})$$

$$R_{3i} \leq E|M_{1i}(\bar{\delta}_i) - M_{1i}(\hat{\delta})|I([A_i^*]^c \cap A_i) + R_{2i} = o(m^{-1}), \quad (\text{A.15})$$

$$R_{4i} \leq E|M_{1i}(\bar{\delta}_i^*) - M_{1i}(\hat{\delta})|I(A_i^* \cap A_i^c) + R_{2i} = o(m^{-1}), \quad (\text{A.16})$$

uniformly in $i \in \{1, \dots, n\}$ (cf. (A.23)-(A.25), Lahiri et al. (2006)). By (A.8), (A.9), and (A.14)-(A.16), the first term on the left of (A.7) is $o(m^{-1})$. The upper bound on the other term on the left of (A.7) follows from condition (C.1), the independence of the resampled vectors (y_1^*, \dots, y_m^*) for $l = 1, \dots, N_0$ and the fact $E_{|\hat{\delta}}(\xi_i(y_i^*; \delta^{*1}) - \xi_i(y_i^*; \hat{\delta}))^2 = M_{2i}(\hat{\delta})$. Hence (A.7) is proved which, in turn, implies that $\max_{1 \leq i \leq m} E|mspe_{\text{NEW}}(\hat{\beta}_i) - mspe_{\text{LM}}(\hat{\beta}_i)| = o(m^{-1})$. Next define the preliminary titled estimator $\bar{\delta}_i$ for the LM method by using the bias and the variance estimators $\hat{b} = b(\hat{\delta})$ and $\hat{V} = V(\hat{\delta})$. Note that with this choice of \hat{b} and \hat{V} , the regularity conditions for the validity of Theorem 3 of LM follow from conditions (C.1)-(C.4) and Lemma 3 above. Hence, (5.4) follows from Theorem 3 of LM.

Appendix B

In this section, the simulation results are presented into subclasses as per the request of a referee. For example, in model I and Model III, the small areas are grouped into 3 classes having equal sampling variances, denoted as G1, G2 and G3. Thus each group represents 5 areas and summary results are presented for each group. But for model II, 3 representative areas are chosen, namely the areas for $n_i = 6$, $n_i = 16$ and $n_i = 36$. Though they are not group in a true sense, they are also represented as G1, G2 and G3 in the tables for convenience. Note that, in this case the estimates represent only these selected three areas, not the averages.

The Table 1b represents the simulated bias and MSPE. For model I, the third group has higher bias and vice versa for model III. For model II, G3, the highest sample size has lowest bias. In terms of MSPE, for all the models, G1 is the highest, although the results between the groups are not drastically different. Also the kernel based method and the closed form formulas (wherever applicable) perform equally.

Table 1b. *Absolute relative bias (T_1) and empirical MSPE (T_2) for the EBP. Results using the kernel based approximations (3.6) and (3.7) are reported within the parentheses.*

Measures	Group	Model I		Model II	Model III	
		Median	Mean		Median	Mean
T_1	G1	2.201	2.087	0.276	—	—
		(2.119)	(2.005)	(0.271)	(1.821)	(1.194)
	G2	1.804	2.196	0.197	—	—
		(2.030)	(2.066)	(0.199)	(1.001)	(1.034)
	G3	2.476	4.846	0.156	—	—
		(2.631)	(4.265)	(0.155)	(.840)	(0.825)
T_2	G1	0.456	0.435	0.015	—	—
		(0.468)	(0.483)	(0.019)	(0.300)	(0.298)
	G2	0.372	0.360	0.013	—	—
		(0.375)	(0.362)	(0.012)	(0.272)	(0.282)
	G3	0.234	0.240	0.003	—	—
		(0.244)	(0.243)	(0.003)	(0.250)	(0.245)

The relative bias and the coefficient of variations of the MSPE estimates are presented in Table 2b. The results for LM1 and LM2 are based on Remark 3 modification. However, they are fairly close when (4.6) was used instead. For all the groups the JLW shows slightly higher bias and CV compared to others. LM1 and PR/DL performs equally well both in terms of bias and CV, LM2 has little higher CV for model I. For model II, CV under JLW is higher than that under LM2. For model III, LM2 performs well for all the groups. For large sample size, the CV under JLW is small yet larger than other methods.

Table 2b. *Relative biases (T_3) and coefficient of variations (T_4) for the bias corrected estimators of the MSPE. Entries for LM1 and LM2 are based on Remark 3 modification.*

Method	Measures	Group	Model I		Model II	Model III	
			Median	Mean		Median	Mean
PR/DL	T_3	G1	0.016	0.090	—	—	—
		G2	0.008	0.063	—	—	—
		G3	0.106	0.084	—	—	—
	T_4	G1	0.184	0.252	—	—	—
		G2	0.151	0.203	—	—	—
		G3	0.119	0.113	—	—	—
JK	T_3	G1	0.287	0.243	-0.190	—	—
		G2	0.124	0.152	-0.083	—	—
		G3	0.124	0.173	0.025	—	—
	T_4	G1	0.924	0.705	1.532	—	—
		G2	0.379	0.449	0.752	—	—
		G3	0.366	0.419	0.360	—	—
LM1	T_3	G1	-0.000	0.072	—	—	—
		G2	-0.017	0.036	—	—	—
		G3	0.061	0.041	—	—	—
	T_4	G1	0.190	0.246	—	—	—
		G2	0.163	0.196	—	—	—
		G3	0.083	0.095	—	—	—
LM2	T_3	G1	-0.093	-0.018	-0.148	-0.005	-0.000
		G2	-0.102	-0.039	0.094	0.102	0.009
		G3	0.003	-0.016	0.054	0.152	0.108
	T_4	G1	0.276	0.300	0.154	0.414	0.368
		G2	0.263	0.274	0.746	0.102	0.009
		G3	0.201	0.202	0.054	0.309	0.202